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Coherent States and Partition Function

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Abstract

The concept of coherent states for arbitrary Lie group is suggested as a tool for explicitly obtaining an integral representation of the partition function, whenever the Hamiltonian has a dynamical group. Two examples are thoroughly discussed: the case of the nilpotent group of Weyl related to a generic many-body problem with two-body interactions, and the case of $\prod_{k\otimes} SU(1, 1)(k)$ relevant for a superfluid system.

1. *Introduction*

Several important properties of coherent states make them ideal for the description of a system with infinitely many degrees of freedom, in which quantum features are macroscopically relevant. They evolve according to classic equations of motion and are, therefore, the most suitable ground for the picture of a system in which low energy excitations are superimposed on a macroscopically occupied ground state which exploits a sort of quasi-classic behavior. Moreover, they constitute a set of functional representatives of the abstract state vector of the system such that every member of this set—translationally invariant in the representation space—is an entire analytic function.

These states are in close connection through Bose statistics and, through the usual commutation relations of second-quantized field theoretical creation and annihilation operators, to the well-known nilpotent Weyl group.

A. M. Perelomov (1972) and M. Rasetti (1973) generalize the concept to different groups defining a set of coherent states for any Lie group, invariant relative to the action of the group generators. These states are determined by a set of points in a suitably defined homogeneous space, and form an over-complete system which contains subsystems of complete coherent states.

As pointed out in Rasetti (1973), the requirement that there be a Lie algebra commuting with the Hamiltonian and that the Lie algebra be integrable to a Lie group, enables us to define and use the coherent states as "spectrum generating" states. It is just this feature to make coherent states interesting, whenever one is in possession of information on the Hamiltonian formulation of a theory, in the sense of its connection with the structure of the underlying dynamical group. In particular, these states may be used as the basis for obtaining a simple representation of the partition function for the given Hamiltonian, as an integral over the group manifold.

Thermodynamic properties of the system, and possible singularities or mathematical pathologies of the partition function, can then be viewed in the perspective of the group manifold topology. In Section 2, we explicit the calculation in the case of nilpotent Weyl group coherent states, essentially on the lines of J. S. Langer (1968). The dynamical group is in this case a broken symmetry, but a perturbative procedure is yet possible. In Section 3, the generalized procedure for both compact and noncompact Lie group coherent states is developed. Section 4 is devoted to the discussion of another case of particular interest: namely $G \equiv \prod_{k\otimes} SU(1,1)_{(k)}$ suitable (Solomon, 1971) for multilevel superfluid Bose system. Section 5 finally concludes with a brief discussion about the analiticity properties of the partition function Z over the symmetric space associated with the coherent state.

2. *The Special Nilpotent Weyl Group*

The Lie algebra of this group is isomorphic to the Lie algebra produced by the usual Bose creation and annihilation operators $a_k^\dagger, a_k$ through the commutation relations

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \tag{2.1}$$

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \tag{2.2}$$

The usual Hamiltonian with two-body interactions can be used without loss of generality (we label the single particle states with the same symbol as for the momentum)

$$H = \sum_k \tilde{\epsilon}_k a_k^\dagger + \frac{1}{2\Omega} \sum_{k,p,q} V(k)\, a_{p+k}^\dagger\, a_{q-k}^\dagger\, a_p a_q \tag{2.3}$$

where the single particle energy ϵ_k has to be modified in order to account for the lack of the number of particles conservation by the introduction of the chemical potential μ:

$$\tilde{\epsilon}_k = \epsilon_k - \mu \tag{2.4}$$

and Ω is the volume of the system.

The operators of the irreducible unitary representation of the Weyl group are of the form

$$\mathrm{T} = e^{i\phi} Д(z) \tag{2.5}$$

where ϕ is a real number, $z \equiv (z_1, z_2, \ldots z_N)$ is an N-dimensional complex vector—N is the number of levels of the system—and

$$Д(z)Д(z') = e^{i\,\mathrm{Im}(z\bar{z}')}Д(z + z') \tag{2.6}$$

If $|\,0\,\rangle$ is the "vacuum" vector defined by the set of equations

$$a_k\,|\,0\,\rangle = 0 \tag{2.7}$$

and it is chosen to be the stationary point in the Hilbert space of the system relative the action of the subgroup generated by the exponential mapping of the identity operator, the system of coherent states is the set of vectors (Klauder, 1970)

$$|\,\{z\}\,\rangle = \prod_k |\,z_k\,\rangle = e^{\sum_k z_k a_k^\dagger}\,|\,0\,\rangle \tag{2.8}$$

It possesses the completeness property

$$\int \prod_k \frac{1}{\pi}\, d\,\mathrm{Re}\, z_k d\,\mathrm{Im}\, z_k\,|\,\{z\}\,\rangle\langle\,\{z\}\,| = I \tag{2.9}$$

The partition function for H may then be written

$$Z = \mathrm{Tr}[e^{-\beta H}] = \int d\mu\,\{z\}\,\langle\,\{z\}\,|\,e^{-\beta H}\,|\,\{z\}\,\rangle \tag{2.10}$$

where

$$d\mu\{z\} = \prod_k \frac{1}{\pi}\, d\,\mathrm{Re}\, z_k d\,\mathrm{Im}\, z_k \tag{2.11}$$

is the invariant measure over the homogeneous space associated with the coherent state system. Explicit evaluation of the integrand is possible, recalling that

$$a_k\,|\,z_k\,\rangle = z_k\,|\,z_k\,\rangle; \quad \langle\,z_k\,|\,a_k^\dagger = \bar{z}_k\langle\,z_k\,| \tag{2.12}$$

In general the matrix element $\langle\,\{z\}\,|\,e^{-\beta H}\,|\,\{z'\}\,\rangle$ can be in fact expanded diagrammatically.

Introducing a free-energy functional $F\{\bar{z}, z'\}$:

$$\langle\,\{z\}\,|\,e^{-\beta H}\,|\,\{z'\}\,\rangle = \langle\,\{z\}\,|\,\{z'\}\,\rangle\, e^{-\beta F\{\bar{z}, z'\}} \tag{2.13}$$

the expansion is equivalent to giving F as a power series of $z_k, z'_{k'}$ (Dyson, 1956):

$$F\{\bar{z}, z'\} = F^{(0)}\{\bar{z}, z'\} + \sum_{n=2}^{\infty} F^{(n)}\{\bar{z}, z'\} \tag{2.14}$$

where

$$F^{(0)}\{\bar{z}, z'\} = -\frac{1}{\beta}\sum_k (e^{-\beta\tilde{\epsilon}_k} - 1)\,\bar{z}_k z'_k \tag{2.15}$$

and

$$F^{(n)}\{\bar{z}, z'\} = \frac{1}{\beta}\sum_{\{k\},\ \{k'\}} W_n(k_1, k_2, \ldots, k_n; k'_1, k'_2, \ldots, k'_n)$$

$$\bar{z}_{k_1}\bar{z}_{k_2}\cdots\bar{z}_{k_n} z'_{k'_1} z'_{k'_2}\cdots z'_{k'_n}\,\delta\Big(\sum_{i=1}^{n}\vec{k}_i - \sum_{i=1}^{n}\vec{k}'_i\Big) \tag{2.16}$$

The delta function is explicitly written because the potential conserves the total momentum. Each W_n is given by the sum of all diagrams with n particle lines.

In the present case, the nilpotent algebra, strictly speaking, does not correspond to a dynamical group of the Hamiltonian, but—according to the prescription of Rasetti (1973)—to a fixed order in $|z|$ it commutes with H, and may be integrated to form the group.

There is a small breaking of the translational invariance in the representation space, which—due to the mentioned global property—can be handled in a perturbative way.

For our purposes of exemplification the only term we are to be concerned with in the following is the two body term $W_2(k_1, k_2; k'_1, k'_2)$. Its diagrammatic expansion is obviously given by the series in Figure 1 or the equivalent integral equation in Figure 2.

The equation of the partition function is now simplified by noting that the total number of excitations, at low temperatures, is small compared to the number of particles; thus for $k_B T \ll \epsilon_k$, $\langle a_k^\dagger a_k \rangle$ must be small, and since $\langle a_k^\dagger a_k \rangle \sim |z_k|^2$ the main correction to $Z^{(0)}$ (the partition function in absence of interaction) comes from small values of z_k.

Consequently one expands the interaction part of the integrand about the origin in z-space.

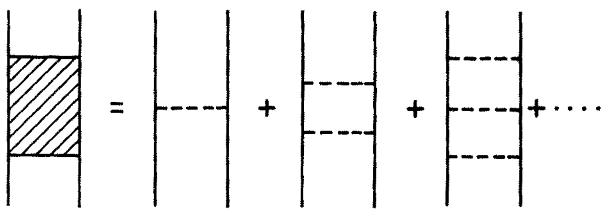


Figure 1

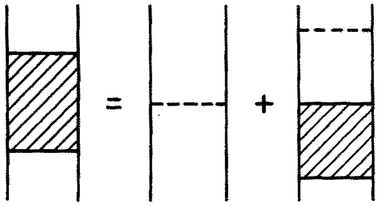


Figure 2

Denoting by $F\{z\}$ the diagonal part of the functional, $F\{z, z\}$, one gets:

$$
\begin{aligned}
Z &= \int d\mu\{z\}\, e^{-\beta F\{z\}} = \int d\mu\{z\}\, e^{-\beta F^{(0)}\{z\}}\, e^{-\beta F^{(2)}\{z\}} \ldots \\
&= \int d\mu\{z\}\, e^{-\beta F^{(0)}\{z\}} \Big[1 - \sum_{\{k\}} W_2(k_1, k_2; k_1', k_2')\, \bar{z}_{k_1} \bar{z}_{k_2} \\
&\qquad \cdot z_{k_1'} z_{k_2'}\, \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_1' - \vec{k}_2') + \cdots \Big] \\
&= Z^{(0)} - \sum_{\{k\}} W_2(k_1, k_2; k_1', k_2')\, \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_1' - \vec{k}_2') \\
&\qquad \cdot \int d\mu\{z\}\, e^{-\beta F^{(0)}\{z\}}\, \bar{z}_{k_1} \bar{z}_{k_2} z_{k_1'}' z_{k_2'}' + \cdots
\end{aligned}
\tag{2.17}
$$

where use has been made of the overlap integral

$$
\langle \{z\} \mid \{z'\} \rangle = \exp\{ - \sum_k [\tfrac{1}{2} | z_k - z_k' |^2 - i\, \mathrm{Im}(\bar{z}_k z_k')]\} \tag{2.18}
$$

and of course only terms involving just two particle diagrams have been kept. It is obviously straightforward to extend the procedure to include higher-order terms, but this does not affect the generality of the description of the method.

It is convenient to introduce "polar" coordinates

$$
z_k = | z_k | \exp(i\theta_k) \tag{2.19}
$$

and one has

$$
d\, \mathrm{Re}\, z_k d\, \mathrm{Im}\, z_k = | z_k | \, d \, | z_k | \, d\theta_k = \tfrac{1}{2} dr_k d\theta_k \tag{2.20}
$$

where $r_k = | z_k |^2$

The integral part of equation (2.17) becomes then

$$
\begin{aligned}
&\int d\mu\{z\}\, e^{-\beta F^{(0)}\{z\}}\, \bar{z}_{k_1} \bar{z}_{k_2} z_{k_3} z_{k_4} \\
&= \left[\prod_{i=1}^{4} \frac{1}{2\pi} \int_0^\infty r_{k_i}^{\frac{1}{2}} dr_{k_i} \int_0^{2\pi} d\theta_{k_i} \right] \int d\mu\{z'\}\, e^{-\beta F^{(0)}\{z\}}\, e^{i\Phi\{\theta_{k_i}\}}
\end{aligned}
\tag{2.21}
$$

where $\{z'\}$ is the complementary set to $\{z_{k_1}, z_{k_2}, z_{k_3}, z_{k_4}\}$ in $\{z\}$, and $\Phi\{\theta_{k_i}\} = \theta_{k_3} + \theta_{k_4} - \theta_{k_1} - \theta_{k_2}$. Note that because of the factor $e^{i\Phi\{\theta_{k_i}\}}$ the integral will vanish unless the total phase $\Phi\{\theta_{k_i}\}$ vanishes. Together with the momentum conservation delta function in front, this leaves only two choices; either $\vec{k}_1 = \vec{k}_3$ or $\vec{k}_2 = \vec{k}_4$, giving obviously identical contributions to W_2. For higher orders the same mechanism would originate a sum over the permutations of the external particle lines.

There are, therefore, only two independent momenta left in the definition of the partition function

$$Z = Z^{(0)} - 2 \sum_{k_1, k_2} W_2(k_1, k_2; k_1, k_2) \int d\mu\{z''\} \cdot [\prod_{i=1}^{2} \int_0^\infty r_{k_i} dr_{k_i}] e^{-\beta F^{(0)}\{z\}} \tag{2.22}$$

where the phase integrals have been executed, each of them giving a factor 2π, and $\{z''\} \cup \{z_{k_1}, z_{k_2}\} = \{z\}$. Recalling the form of $F^{(0)}\{z\}$, integrals over $d\mu\{z''\}$ give each a factor

$$\frac{1}{\pi} \int_0^\infty e^{-\bar{z}_{k_i} z_{k_i}(1-e^{-\beta\tilde{\epsilon}_{k_i}})} d\,\mathrm{Re}\, z_{k_i} d\,\mathrm{Im}\, z_{k_i} = \frac{1}{1 - e^{-\beta\tilde{\epsilon}_{k_i}}}, \qquad i \neq 1, 2 \tag{2.23}$$

The remaining two integrals are

$$\int_0^\infty e^{-r_{k_i}(1-e^{-\beta\tilde{\epsilon}_{k_i}})} \cdot r_{k_i} dr_{k_i} = \frac{1}{(1 - e^{-\beta\tilde{\epsilon}_{k_i}})^2}, \qquad i = 1, 2 \tag{2.24}$$

Thus, there is a factor of $(1 - e^{-\beta\tilde{\epsilon}_{k_i}})^{-1}$ for all k_i, and the product of all of these is just the partition function for non interacting particles $Z^{(0)}$. Z becomes then

$$Z = Z^{(0)} [1 - 2 \sum_{k_1, k_2} \frac{1}{(1 - e^{-\beta\tilde{\epsilon}_{k_1}})(1 - e^{-\beta\tilde{\epsilon}_{k_2}})} W_2(k_1, k_2; k_1, k_2) + \cdots] \tag{2.25}$$

And the free energy

$$F = -\frac{1}{\beta} \lg Z \approx F^{(0)} + \frac{2}{\beta} \sum_{k_1, k_2} \frac{1}{(1 - e^{-\beta\tilde{\epsilon}_{k_1}})(1 - e^{-\beta\tilde{\epsilon}_{k_2}})} W_2(k_1, k_2; k_1, k_2) + \cdots \tag{2.26}$$

The second factor is the leading dynamical correction to the free energy. Other factors would of course be recovered by expanding $F\{z\}$ to higher orders about the origin in z-space. Now only the explicit calculation of W_2 is left over in order to evaluate both the partition function and the free energy dynamically corrected. The calculation is standard, and it is shortly discussed in the following for sake of completeness, even if it somewhat exorbits in its spirit the content of the present paper.

First an integral equation for W_2 should be written consisting of the formal sum of all the diagrams. Such an equation, however, depends on the temperature in a complicated way which makes it rather different from the ordinary scattering equation. Observing that temperature enters each diagram only through integrals over Boltzmann factors, which are indeed convolution integral with finite upper limit (the energy is the sum of all unperturbed single particle energies), the use of the "temperature" Laplace transform allows us to relate the equation for W_2 to that of the usual t-matrix (Bloch and De Dominicis, 1958; Abrikosov, Gorkov, and Dzialoshinski, 1963).

The transformation can be performed over each diagram separately, and this amounts to a change in the rules for evaluating the diagram itself.

Consider Figure 3. Its value, by the usual rules is

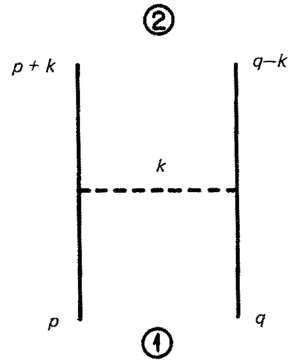


Figure 3

$$\frac{1}{\Omega} V(k) \int_0^\beta e^{-\beta' E_1} e^{-(\beta-\beta')E_2}\, d\beta' = \frac{1}{\Omega} V(k) \frac{e^{-\beta E_1} - e^{-\beta E_2}}{E_1 - E_2} \tag{2.27}$$

where the energy of the two free particles is, e.g.,

$$E_1 = \tilde{\epsilon}_p + \tilde{\epsilon}_q = \epsilon_p + \epsilon_q - 2\mu \tag{2.28}$$

$$E_2 = \epsilon_{p+k} + \epsilon_{q-k} - 2\mu \tag{2.29}$$

Since the total momentum of the pair $\vec{P} = \vec{p} + \vec{q}$ is a constant of the motion and will be the same in each term, it is convenient to label the two configurations ① and ② by half the relative momentum

$$\vec{Q} = \tfrac{1}{2}(\vec{p} - \vec{q}) \qquad \vec{Q}' = \tfrac{1}{2}[(\vec{p} + \vec{k}) - (\vec{q} - \vec{k})] = \vec{Q} + \vec{k} \tag{2.30}$$

The Laplace transform of the diagram then reads

$$\int_0^\infty e^{-\beta s} \frac{1}{\Omega} V(k) \frac{e^{-\beta E_Q} - e^{-\beta E_{Q'}}}{E_Q - E_{Q'}}\, d\beta = \frac{1}{E_{Q'} + s} \frac{V(k)}{\Omega} \frac{1}{E_Q + s} \tag{2.31}$$

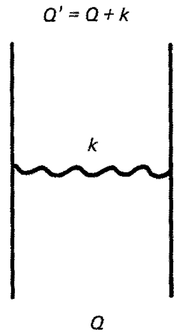


Figure 4

The Laplace transform of a convolution is just the product of individual Laplace transforms.

By comparison of this result with the diagram, one extracts the following set of rules (in the graphical representation we will use wavy lines for the interaction, in order to distinguish the two).

Given Figure 4, one has

(a) a factor $V(k)/\Omega$ for each interaction line;
(b) a factor $1/(E+s) = \int_0^\infty e^{-\beta(E+s)}\, d\beta$ for each horizontal line which may be drawn between successive interactions;
(c) symmetry factors and sum over internal momenta as usual.

In this way one may define the Laplace transform of W_2 as

$$\int_0^\infty e^{-\beta s} W_2(\vec{p},\vec{q};\vec{k}+\vec{p},\vec{q}-\vec{k})\, d\beta = \frac{1}{E_{Q'}+s}\,\frac{t(Q,Q')}{\Omega}\,\frac{1}{E_Q+s} \tag{2.32}$$

and the matrix $t(Q, Q')$ turns out to be the usual t-matrix for the scattering of two bosons of total energy $E = -s$. Indeed, it satisfies the integral equation (Figure 5) which, written explicitly reads:

$$t(Q,Q') = V(k) - \frac{2}{\Omega}\sum_{Q''} V(Q'-Q'')\,\frac{1}{E_{Q''}+s}\,t(Q,Q'') \tag{2.33}$$

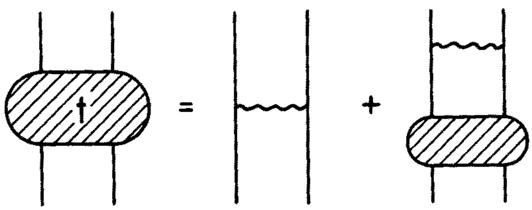


Figure 5

Once such an equation is solved, W_2 is simply determined by computing the inverse transform

$$W_2(\vec{p}, \vec{q}; \vec{p} + \vec{k}, \vec{q} - \vec{k}) = \frac{1}{2\pi i \Omega} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{1}{E_{Q'} + s} t(Q, Q') \frac{1}{E_Q + s} e^{\beta s} ds \tag{2.34}$$

where the integration is over the usual Brömwich contour defined by s_0 to the right of any singularities in the s-plane of the integrand.

In other words, the leading dynamical correction to the partition function is related to the inverse Laplace transform of $t(Q, Q)$.

Now it is reasonable to assume that $t(Q, Q')$ is analytic in s except for the poles corresponding to the bound states of two bosons, and possibly along a branch cut, i.e., a densely discrete set of poles corresponding to a band.

In such a situation, the contour of integration for the integral defining W_2 may be deformed and the integral evaluated as the sum over the residues plus the integral around the branch cut.

One gets

$$W_2(\vec{p}, \vec{q}; \vec{p} + \vec{k}, \vec{q} - \vec{k}) = \frac{t(Q, Q')\,|_{s=-E_Q} e^{-\beta E_Q} - t(Q, Q')\,|_{s=-E_Q} e^{-\beta E_{Q'}}}{E_Q - E_{Q'}}$$

$$+ \frac{1}{\pi} \int_{s_-}^{s_+} \frac{1}{E_{Q'} + s} \operatorname{Im} t\,(Q, Q')\,|_{s - i\epsilon} \frac{1}{E_Q + s} e^{\beta s} ds \tag{2.35}$$

where the branch cut lies along the axis between s_- and s_+ and Im t is to be evaluated below the cut.

Generally, the branch cut integration leads to a negligible factor in the sense that it is of higher order in the temperature. The diagonal element of W_2, which enters the correction to Z can therefore be approximated as

$$W_2(p, q; p, q) \approx \frac{1}{\Omega} \beta\, e^{-\beta E_Q}\, t(Q, Q)\,|_{s=-E_Q} \tag{2.36}$$

3. *The General Case*

The generalized definition for coherent states of an arbitrary Lie group G is the following. Let T be the irreducible unitary representation of G acting in the Hilbert space $\mathscr{H}$; $|\,\Phi_0\rangle$ some fixed vector in $\mathscr{H}$; $\hbar$ the stationary subgroup of G with respect to $|\,\Phi_0\rangle$ and $M = G/\hbar$ the homogeneous factor space. Then a system of coherent states is the set $\{|\,\Phi_g\rangle\}$

$$|\,\Phi_g\rangle = \mathrm{T}(g)\,|\,\Phi_0\rangle, \qquad g \in G \tag{3.1}$$

where g runs over the whole group G. The coherent state $|\,\Phi_g\rangle$ is then uniquely determined by the point $x = x(g), x \in M$

$$|\,\Phi_g\rangle = e^{i\alpha(g)}\,|\,x\rangle;\ |\,\Phi_0\rangle = |\,0\rangle \tag{3.2}$$

In (3.2) $e^{i\alpha(h)}$ is a one dimensional unitary representation of $\not{h}$, for $h \in \not{h}$; the function $\alpha(g)$ on the other hand is determined on the entire group G and for $g \in \not{h} \subset G$, it coincides with $\alpha(h)$. It can be easily seen that the set of coherent states is invariant relative to the action of the group representation operators:

$$\mathrm{T}(g') \mid \Phi_g \rangle = \mathrm{T}(g')\, \mathrm{T}(g) \mid \Phi_0 \rangle = \mathrm{T}(g'') \mid \Phi_0 \rangle = \mid \Phi_{g''} \rangle \tag{3.3}$$

where

$$g'' = g'g \in G \tag{3.4}$$

This formula can be rewritten

$$\begin{aligned} \mathrm{T}(g') \mid x \rangle &= e^{-i\alpha(g)}\, \mathrm{T}(g') \mid \Phi_g \rangle = e^{-i\alpha(g)} \mid \Phi_{g''} \rangle \\ &= e^{-i\alpha(g)}\, e^{i\alpha(g'')} \mid x'' \rangle = e^{i\beta(g,g')} \mid g'x \rangle \end{aligned} \tag{3.5}$$

with

$$\beta(g, g') = \alpha(g'g) - \alpha(g) \tag{3.6}$$

showing that the complete set of coherent states is actually generated by the action of the group G on the homogeneous space M. This isomorphism between $\mathscr{H}$, and a set $\{f\}$ of analytic functions over the manifold M, is of special physical interest in the case when the functions are square-integrable. The system of coherent states can then be defined whenever the integral

$$\int | \langle 0 \mid x \rangle |^2\, d_\mu x = \Gamma \tag{3.7}$$

where $d_\mu x$ denotes a "measure" on M, converges, i.e., Γ is a finite constant. In such a case

$$\frac{1}{\Gamma} \int d_\mu x \mid x \rangle \langle x \mid = I \tag{3.8}$$

and one can expand any arbitrary state in coherent states.

Moreover on M one has the important set of two points amplitudes ("reproducing kernels")

$$K(x, y) = \frac{1}{\Gamma} \langle x \mid y \rangle \tag{3.9}$$

which satisfy the integral equation

$$K(x, z) = \int d_\mu y\, K(x, y)\, K(y, z) \tag{3.10}$$

i.e., K reproduces itself under convolution, and for an arbitrarily chosen function $f \in \{f\}$, gives rise to the integral identity

$$f(x) = \frac{1}{\Gamma} \int d_\mu y \langle x \mid y \rangle f(y) \tag{3.11}$$

This is equivalent to performing a harmonic analysis on M in terms of irreducible representations of G (compare for example the solution of Helmholtz equation in terms of prolate spheroidal coordinates in R^3). So we may think of K as the kernel of a linear transformation in M. Due to the general differential geometric versus algebraic theoretical approach, the method is valid for both compact and noncompact groups.

The partition of identity, which depends on the overcompleteness of the set of coherent states system (this means that there are subsystems still being coherent and complete states), allows us to write the partition function Z as

$$Z = \int_{M^{\#}} d_\mu x \langle x \mid e^{-\beta H} \mid x \rangle = \lim_{n' \to \infty} \int_{M^{\#}} d_\mu x \langle x \mid (I - \frac{\beta}{n'} H)^{n'} \mid x \rangle \tag{3.12}$$

$\mid x \rangle$ being the coherent states for some Lie group G, and $M^{\#} \subseteq M$.

The group G is now assumed not to be a symmetry group of the Hamiltonian H, but its dynamical group, i.e., a group whose algebra may be used to generate the spectrum of H. In this case the dynamical problem involves a Hamiltonian which may in general be expressed in terms of a set of operators which are generators of the algebra g of G. In the simplest case H is a linear combination of the dynamical group generators (possibly a direct sum if the spectrum generating group is a direct product of subgroups $G^{(k)}$)

$$H = \sum_{j=1}^{n} \oplus_k \, \omega_j J_j^{(k)} \tag{3.13}$$

where the index k runs over the set of labels of $G^{(k)}$, and denoting by α_{ijl} the structural constant of G, g is defined by

$$[J_i^{(k)}, J_j^{(k')}] = \alpha_{ijl} J_l^{(k)} \delta_{k',k} \tag{3.14}$$

(summation is implied on the dummy indexes).

A general element of the Lie algebra can be written as

$$S = i(\bar{g} J - g J^{\dagger}) \tag{3.15}$$

where

$$\bar{g}J = \bar{g}_i J_i, \qquad gJ^{\dagger} = g_i J_i^{\dagger} \tag{3.16}$$

$g \equiv (g_1, \ldots, g_n)$ being an n-dimensional complex vector. The Lie group is obtained from the algebra by the usual exponential mapping: the operator in (3.12) is just a representative $T(g_0)$ of a selected $g_0 \in G$ (in general $g_0 = \prod_k \otimes g_0^{(k)}$).

Differential operators on $\mathscr{H}$ corresponding to the infinitesimal generators $J_i^{(k)}$ may be obtained by standardized procedure, and finally all the irreducible representations can be found by considering the action of such generators on the monomials in $\mathscr{H}$. Unitary irreducible representations are constructed by suitably normalizing the basis vectors, introducing an inner product and imposing the proper hermiticity condition on the generators.

The orthonormal basis vectors may typically be labelled by the eigenvalues of the Casimir operators $C^{(k)}$ and of one of the infinitesimal generators, say $J_\alpha^{(k)}$. (Indeed they are eigenvectors of $J_\alpha^{(k)}$ in the representation corresponding to the subgroup h.)

In principle a rotation may be performed in the space of the algebra about some $J_\beta^{(k)}$ axs—which is just a generalization of the Bogolubov transformation (Solomon, 1971)

$$R(\theta_k) = \exp[-iJ_\beta^{(k)}\theta_k] \tag{3.17}$$

$$R = \prod_{k\otimes} R^{(k)}(\theta_k) \tag{3.18}$$

and the set $\{\theta_k\}$ be chosen in such a way that the rotated Hamiltonian

$$H_R = RHR^{-1} = \sum_{k\otimes} \tilde{\omega}_k J_\alpha^{(k)} \tag{3.19}$$

depends—linearly—only on the diagonalized generator.

The eigenvalues of the Hamiltonian are then trivially dependent on the eigenvalues of $J_\alpha^{(k)}$. That is the reason why the group G is referred to as energy spectrum generating group, even though it is not itself a symmetry group of the Hamiltonian.

According to the general definition the set of coherent states are points of the manifold M, on which the action of the group is given by a differentiable mapping

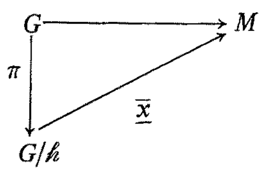
$$f : G \times M \to M : (gx) \to f(x, g) \tag{3.20}$$

For a fixed $x_0 \in M$, the corresponding coherent state is determined by the mapping

$$\underline{x}(g) = f(x, g) \tag{3.21}$$

such that $\underline{x}^{-1}(x_0) = h$; where h is the closed subgroup of G which leaves x_0 fixed. Since h is closed it is again a Lie group (Rasetti, 1973) and G/h is an analytic manifold.

The map $\underline{x}$ induces a map $\overline{\underline{x}} : G/h \to M : (g,h) \to f(g, x)$. This mapping is well defined since h leaves the point x_0 fixed and the following diagram commutes



where π denotes the natural projection $g \to gh$ of G onto G/h.

It follows that the orbits of G on M are submanifolds of M. Complete coherent states constitute therefore a set $M^{\#}$ of orbits in M. On the other hand, in our case H_R is nothing but J_α and its action over $|x\rangle$ induces a flow on M.

$$\langle x' \mid e^{-\beta H_R} \mid x\rangle = \prod_{k\otimes} \langle x' \mid \mathrm{T}(g_0^{(k)}) \mid x\rangle = \sum_k \langle x' \mid g_0^{(k)} x\rangle \tag{3.22}$$

where $\mathrm{T}(g_0^{(k)})$ is the representative of the group translation induced on M by the rotated Hamiltonian H_R.

Note that since the integral in the definition of Z is over the manifold M, it is invariant under the substitution of H with H_R.

It was already pointed out that

$$\langle x' \mid x''\rangle = e^{i[\alpha(g') - \alpha(g'')]} \langle 0 \mid \mathrm{T}(g'^{-1} g'') \mid 0\rangle = \overline{\langle x'' \mid x'\rangle} \tag{3.23}$$

and therefore

$$\langle x' \mid g_0^{(k)} x''\rangle = e^{i[\alpha(g') - \alpha(g'') - \beta(g_0^{(k)}, g'')]} \langle 0 \mid \mathrm{T}(g'^{-1} g_0^{(k)} g'') \mid 0\rangle \tag{3.24}$$

It follows that evaluation of Z is nothing but an integration (indeed over the submanifold $M^{\#}$ of all the group orbits) of the scalar product of the generic coherent state with the fixed vector $|0\rangle$.

It is to be noted how, even in the case when H is not simply linear in the J_i's, the application of Baker-Campbell-Haussdorff formula, which is at the basis of our generalized Bogolubov rotation, together with the commutation relations (3.14) lead to the same structure when G is the spectrum generating group of H. Moreover, the exponential mapping does not have a vanishing Jacobian at the origin (i.e., it is a diffeomorphism of an open neighborhood of zero in g ($g_1 = \cdots = g_n = 0$) onto an open neighborhood of the identity in G) and therefore, any analytic function at the identity can be expanded in some neighborhood of $\mathbf{0} \in g$.

This leads to a diagrammatic expansion very similar to the case of the previous section when the group G is not itself a spectrum generating group (as the Weyl group) but the terms breaking the symmetry are "small." This is very similar to a random phase approximation.

Before closing this section, it is worth observing that the proposed method can be considered as a sort of generalization of the Feynman's path-integral approach to statistical mechanics.

4. *Superfluid Example*

It has been shown by A. I. Solomon (1971) that in the framework of the Foldy model approximation (Bassichis and Foldy, 1964), a many level superfluid system has a spectrum generating group which is a direct product $\prod_{k\otimes} SU(1, 1)_{(k)}$ (where as usual the index k labels both the momenta and the levels).

The Hamiltonian in such a case may be written

$$H = \sum_{k\otimes} NV_k(-J_1^{(k)} + \mu_k J_2^{(k)} - \tfrac{1}{2}\mu_k) + \tfrac{1}{2}N^2 V_0 \tag{4.1}$$

where N is the total number of bosons, ϵ_k the energy of level kth, V_k the Fourier transform of the interaction potential and

$$\mu_k = 1 + \frac{\epsilon_k}{NV_k} \tag{4.2}$$

$J_i^{(k)}$ ($i = 1, 2, 3$) are the generators of $SU(1, 1)_{(k)}$, such that

$$[J_1^{(k)}, J_2^{(k')}] = -iJ_3^{(k)}\delta_{kk'}; \qquad [J_2^{(k)}, J_3^{(k')}] = iJ_1^{(k)}\delta_{kk'};$$
$$[J_3^{(k)}, J_1^{(k')}] = iJ_2^{(k)}\delta_{kk'} \tag{4.3}$$

By the hyper-rotation

$$R = \prod_{k\otimes} R^{(k)}(\theta_k), \qquad R^{(k)}(\theta_k) = \exp(-J_2^{(k)}\theta_k) \tag{4.4}$$

with

$$\theta_k = \coth^{-1}\mu_k \tag{4.5}$$

one gets

$$H_R = RHR^{-1} = \sum_{k\otimes} (\operatorname{csch}\theta_k J_3^{(k)} - \tfrac{1}{2}\mu_k)NV_k + \tfrac{1}{2}N^2 V_0 \tag{4.6}$$

The system of coherent states $\{ | \xi \rangle \}$ for the universal covering group of $SU(1, 1)$ has been thoroughly discussed by Perelomov (1973). Its relevant properties are briefly reconsidered hereafter.

The factor space M, isomorphic to the upper sheet of a three-dimensional hyperboloid, is the unit disk $|\xi| < 1$, whose invariant Riemannian measure is

$$d\mu(\xi) = \frac{d\,\mathrm{Re}\,\xi\, d\,\mathrm{Im}\,\xi}{(1 - |\xi|^2)^2} \tag{4.7}$$

The scalar product is given by

$$\langle \xi | \xi' \rangle = (1 - |\xi'|^2)^{\kappa} (1 - |\xi|^2)^{\kappa} (1 - \bar{\xi}'\xi)^{-2\kappa} \tag{4.8}$$

where κ is an arbitrary nonnegative number, and the condition of completeness has the form

$$\frac{2\kappa - 1}{\pi} \int d\mu(\xi) \, | \xi \rangle \langle \xi | = I \tag{4.9}$$

Finally, the action of a group representation operator $\mathrm{T}^{(\kappa)}(g)$ on $| \xi \rangle$ results in

$$\mathrm{T}^{(\kappa)}(g) \, | \xi \rangle = e^{i\phi} \, | \xi' \rangle \tag{4.10}$$

where, parametrizing the group by the set of 2×2 complex matrices

$$g = \begin{Vmatrix} a & b \\ \bar{b} & \bar{a} \end{Vmatrix}, \qquad \det g = |a|^2 - |b|^2 = 1 \tag{4.11}$$

one has

$$\xi' = \frac{a\xi - b}{-\bar{b}\xi + \bar{a}}; \qquad \phi = 2\kappa \arg(a - b\xi) \tag{4.12}$$

so that

$$\langle \eta | \mathrm{T}^{(\kappa)}(g) | \zeta \rangle = e^{i\phi} \langle \eta | g\zeta \rangle = (1 - |\eta|^2)^{\kappa}(1 - |\zeta|^2)^{\kappa}(\bar{a} - \bar{b}\zeta + b\bar{\eta} - a\bar{\eta}\zeta)^{-2\kappa} \tag{4.13}$$

and

$$\langle \xi | \mathrm{T}^{(\kappa)}(g) | \xi \rangle = (1 - |\xi|^2)^{2\kappa}(\bar{a} - a|\xi|^2 - 2i \operatorname{Im} \bar{b}\xi)^{-2\kappa} \tag{4.14}$$

If the above-mentioned procedure of rotation is performed, the exponentiation of the Hamiltonian leads simply to the irreducible representation of an element h of the subgroup $\mathscr{h}$;

$$h \equiv \begin{Vmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{Vmatrix}$$

and the representation T of G being restricted on the subgroup has to contain, by the Fröbenius reciprocity theorem, the one-dimensional abelian representation of $\mathscr{h}$ itself. Except for a phase factor

$$\frac{a}{\bar{a}} = e^{i\psi}$$

ξ' is identical to ξ:

$$\langle \xi | \mathrm{T}^{(\kappa)}(h) | \xi \rangle = (1 - |\xi|^2)^{2\kappa}(1 - e^{i\psi}|\xi|^2)^{-2\kappa} e^{i\kappa\psi} \tag{4.15}$$

Substituting back into (3.12) one gets

$$Z = \sum_k \left\{ e^{\frac{1}{2}\beta N(\mu_k V_k - N V_0)} \frac{2\kappa - 1}{\pi} e^{\kappa\beta \operatorname{csch} \theta_k} \int_{|\xi|<1} \frac{d \operatorname{Re} \xi \, d \operatorname{Im} \xi (1 - |\xi|^2)^{2\kappa - 2}}{(1 - e^{\beta \operatorname{csch} \theta_k} |\xi|^2)^{2\kappa}} \right\} \tag{4.16}$$

The analysis of Solomon shows that the Casimir operators

$$C_k = -(J_1^{(k)})^2 - (J_2^{(k)})^2 + (J_3^{(k)})^2 \tag{4.17}$$

can be written

$$C_k = j_k(j_k + 1) = \tfrac{1}{4}(\Delta_k^2 - 1) \tag{4.18}$$

where

$$\Delta_k = a_k^\dagger a_k - a_{-k}^\dagger a_{-k} \tag{4.19}$$

are the differences between number of particles in opposite momentum states, so that

$$j_k = -\tfrac{1}{2} - \tfrac{1}{2}\,|\Delta_k| \tag{4.20}$$

On the other hand, the only allowed representation is given by the positive discrete series (Vilenkin, 1968)

$$\prod_k {}_{\otimes}\mathscr{D}^+(j_k), \qquad j_k = -\kappa \tag{4.21}$$

This suggests the choice $\kappa = \frac{1}{2}$.

Such a choice may appear singular at first sight, because of equation (4.9), but the limit $\kappa \to \frac{1}{2}$ indeed exists and it is finite.

Any integral over the unit disk can, in general, be reduced in the form of ordinary integrals performing harmonic analysis (Helgason, 1962, 1965) over the non-Euclidean manifold $D \equiv \{\xi : |\xi| \langle 1\}$. Denote in the following by $B = \partial D$ the boundary of D, and define

$$\langle \xi, b\rangle = \frac{(\xi, b)}{(1 - |\xi|^2)^2}, \qquad \xi \in D, \qquad b \in B \tag{4.22}$$

as the non-Euclidean distance from 0 to the orthogonal trajectory to the family of all parallel geodesics corresponding to $b = e^{i\phi}$, passing through $\xi((\xi, b)$ is the usual inner product in R^2). Consider moreover the Hilbert space

$$\mathscr{H}_\lambda = \{h_\lambda(\xi) = \int_B e^{(i\lambda + 1)\langle \xi, b\rangle} h(b)db \mid h \in \mathscr{L}^2(B)\} \tag{4.23}$$

Then

$$[\mathrm{T}_\lambda(g)h_\lambda](\xi) = h_\lambda(g^{-1}\xi) \tag{4.24}$$

define the unitary irreducible representation of G in $\mathscr{H}_\lambda$. Moreover

$$h_\lambda(g^{-1}\xi) = \int_B e^{(i\lambda+1)\langle \xi, b\rangle} e^{(-i\lambda+1)\langle g.0, b\rangle} h(g^{-1}b)db \tag{4.25}$$

On the other hand

$$P(\xi, b) = e^{2\langle \xi, b\rangle} = \frac{1 - |\xi|^2}{1 - 2|\xi|\cos(\theta - \phi) + |\xi|^2}, \qquad \xi = |\xi|\, e^{i\theta} \tag{4.26}$$

is just the Poisson Kernel expressed in non-Euclidean terms. Therefore recalling the Laplace-Beltrami operator (Karpelevic, 1965) on D

$$\Delta : f \to \frac{1}{\sqrt{\det(g_{ij})}} \sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{\det(g_{ij})}\frac{\partial}{\partial x_i}\right) f \tag{4.27}$$

with

$$g_{ij} = [1 - |\xi|^2]^{-2}\delta_{ij} \tag{4.28}$$

$$g^{ij} = (g_{ij})^{-1}; \qquad \det(g_{ij}) = (1 - |\xi|^2)^{-4} \tag{4.29}$$

where δ_{ij} is the Kronecker delta symbol

$$\Delta_\xi = [1 - |\xi|^2]^2 \left[\frac{\partial^2}{\partial[\mathrm{Re}\,\xi]^2} + \frac{\partial^2}{\partial[\mathrm{Im}\,\xi]^2}\right] \tag{4.30}$$

one easily finds that any power of the Poisson Kernel gives an eigenfunction of the non-Euclidean Laplacian

$$\Delta_\xi [P(\xi, b)]^\mu = 4\mu(\mu - 1) P(\xi, b); \qquad \mu \in \mathbf{C} \tag{4.31}$$

with eigenvalue independent on b.

If $M(B)$ denotes the set of analytic functions on B, which is considered as an analytic manifold (observe that M carries a natural topology), the continuous linear functions $\nu : M(B) \to \mathbf{C}$ constitute a space $M^d(B)$ dual of $M(B)$; ν are called analytic functionals on B.

A theorem by S. Helgason proves that

$$F(\xi) = \int_B P(\xi, b)^\zeta d\nu(b); \qquad \nu \in M^d \tag{4.32}$$

are eigenfunctions of Δ_ξ, with real eigenvalues if $\zeta \in R$.

Hence integration over ζ amounts to performing harmonic analysis over D with respect to G. One may check there exists a measure

$$d\mu(\lambda) = \frac{1}{2\pi^2} \lambda \, th\left(\frac{\pi}{2}\lambda\right) d\lambda$$

such that

$$\int_{R/Z_2} \mathscr{H}_\lambda d\mu(\lambda) = \mathscr{L}^2(D) \tag{4.33}$$

(the integration is over R/Z_2, because of definition (4.23) and equation (4.24) which imply that T_{λ_1} is equivalent to T_{λ_2} only if $\lambda_1 = -\lambda_2$). One has therefore:

$$\int_D f(\xi) d\mu(\xi) = \int_{R \times B} d\mu(\lambda)\, db\, \tilde{f}(\lambda, b)\, \tilde{\Delta}(\lambda, b) \tag{4.34}$$

where

$$\tilde{\Delta}(\lambda, b) = \frac{1}{(2\pi)^2} \int_D e^{(-i\lambda + 1)\langle \xi, b \rangle} d\mu(\xi) \tag{4.35}$$

and

$$\tilde{f}(\lambda, b) = \int_D f(\xi)\, e^{(-i\lambda + 1)\langle \xi, b \rangle} d\mu(\xi) \tag{4.36}$$

being

$$f(\xi) = \frac{1}{(2\pi)^2} \int_{R \times B} \tilde{f}(\lambda, b)\, e^{(i\lambda + 1)\langle \xi, b \rangle} d\mu(\lambda)\, db \tag{4.37}$$

In the present case the calculation is much easier because $f(\xi) = f^{\#}(r)$ is only function of the distance r from 0 to the geodesics through ξ

$$r = \frac{1}{2} \ln \frac{1 + |\xi|}{1 - |\xi|}, \qquad |\xi| = \tanh \tag{4.38}$$

and

$$d\mu(\xi) = \pi \sinh 2r \cdot dr; \qquad 0 \leqslant r < \infty \tag{4.39}$$

Both $\tilde{f}$ and $\tilde{\Delta}$ are function of λ alone

$$\int_B e^{(i\lambda + 1)\langle \xi, b \rangle} db = P_{-\frac{1}{2}-\frac{1}{2}i\lambda}(\cosh r) \cosh (2r) \tag{4.40}$$

and

$$\tilde{\Delta}(\lambda) = \pi \int_0^\infty P_{-\frac{1}{2}-\frac{1}{2}i\lambda}(\cosh r) \cosh (2r) \sinh (2r)\, dr \tag{4.41}$$

while $f^{\#}(r)$ is simply the Mehler transform of $f(\lambda)$; $P_\eta(\cosh r)$ being the Legendre function. A simple calculation and changing of variables allows writing

$$Z = \frac{1}{2\pi} \sum_k \left\{ e^{\frac{1}{2}\beta E_k} \frac{A_k}{\cosh \theta_k} \int_{e^{-2\theta_k}}^{e^{2\theta_k}} \frac{dx}{x} \frac{(x+1)(1 - e^{-\beta E_k} x)}{x^2 + 2A_k x + e^{\beta E_k}} \right. \tag{4.42}$$

where

$$\left. A_k = \frac{\sinh^2 \theta_k}{2 \cosh^2 \theta_k - e^{-\beta E_k}} \right\} \tag{4.43}$$

θ_k being given by equation (4.5) and (4.2), and E_k by the known Bogolubov formula

$$E_k = (\epsilon_k{}^2 + 2\epsilon_k N V_k)^{\frac{1}{2}} \tag{4.44}$$

Use now the integral representation (Vilenkin, 1968)

$$\mathscr{B}^l_{mn}(\cosh \tau) = \frac{1}{2\pi i} \oint_{\Gamma'} w^{l-n} \left(\cosh \frac{\tau}{2} + z \sinh \frac{\tau}{2} \right)^{2n} \frac{z^{m-n}}{\sqrt{w^2 - 2w \cosh \tau + 1}} dw \tag{4.45}$$

where the contour Γ' is denoted in Figure 6, and

$$z = \frac{w - \cosh \tau \pm \sqrt{w^2 - 2w \cosh \tau + 1}}{\sinh \tau}$$

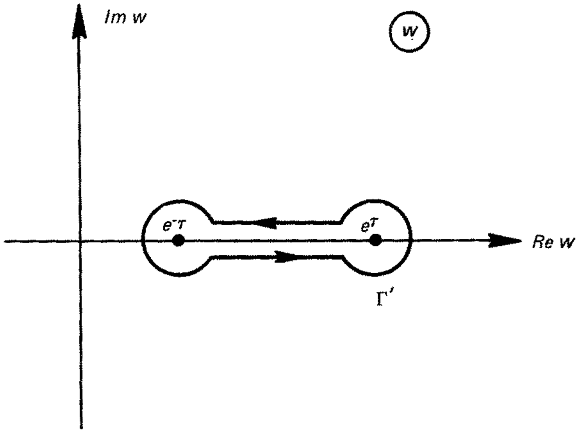


Figure 6

the radical being chosen such that $1 \leqslant |z| \leqslant \coth(\tau/2)$; for the Jacobi function $\mathscr{B}^l_{mn}(\cosh\tau)$, by means of which the matrix elements of the representation

$$\mathrm{T}^{(\kappa)}(g_\tau), \quad g_\tau \equiv \begin{Vmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{Vmatrix} \in \Omega_1 \subset SU(1,1)$$

can be expressed as

$$t^{(j,\sigma)}_{m,n}(g_\tau) = \mathscr{B}^j_{m+\sigma,n+\sigma}(\cosh\tau) \tag{4.46}$$

Contracting the integration contour to the segment which it encloses, assuming $j = -\sigma = -\frac{1}{2}$, one gets

$$\mathscr{B}^{-\frac{1}{2}}_{n+\frac{1}{2},n+\frac{1}{2}}(\cosh\tau) = \frac{1}{\pi}\int_0^\tau \frac{\cosh[(n+\frac{1}{2})t]\,\cos[(2n+1)\alpha]}{\sqrt{\cosh^2\frac{\tau}{2} - \cosh^2\frac{t}{2}}} \tag{4.47}$$

with

$$\cos\alpha = \frac{\cosh\frac{\tau}{2}}{\cosh\frac{\tau}{2}} \tag{4.48}$$

and finally

$$Z = \sum_k e^{-\frac{1}{2}\beta E_k} \sum_{n_k} e^{-\beta n_k E_k} t_{n_k, n_k}^{(-\frac{1}{2}, \frac{1}{2})} (g_0^{(k)}) \tag{4.49}$$

$$g_0^{(k)} \equiv \begin{Vmatrix} \cosh \theta_k & \sinh \theta_k \\ \sinh \theta_k & \cosh \theta_k \end{Vmatrix} \tag{4.50}$$

The use of coherent states gives at once the result which, by use of the explicitly calculated eigenstates of the Hamiltonian (Solomon, 1971)

$$| \psi\{n_k\} \rangle = \prod_{k\otimes} \sum_{m_k} t_{m_k n_k}^{(-\frac{1}{2}, \frac{1}{2})} (g^{(k)}) \, | m_k \rangle \tag{4.51}$$

$$g_0^{(k)} = \begin{Vmatrix} \cosh \dfrac{\theta_k}{2} & \sinh \dfrac{\theta_k}{2} \\ \sinh \dfrac{\theta_k}{2} & \cosh \dfrac{\theta_k}{2} \end{Vmatrix} \tag{4.52}$$

could have been obtained—in the relatively simple example here examined—by direct computation.

The above discussed method is, however, more powerful. Indeed, recalling the general relation between H and G (Rasetti, 1973), and the characteristic property of coherent states (Perelomov, 1973)

$$| \xi \rangle = (1 - | \xi |^2)^{\kappa} e^{\xi \sum_{k\oplus} J_+^{(k)}} \, | 0 \rangle \tag{4.53}$$

and

$$\xi J_+^{(k)} \, | \xi \rangle = (J_3^{(k)} - \kappa) \, | \xi \rangle \tag{4.54}$$

$$J_-^{(k)} \, | \xi \rangle = \xi (J_3^{(k)} + \kappa) \, | \xi \rangle \tag{4.55}$$

even in the case when the Hamiltonian is not simply linear in the group generators one can—exactly in the same way as in Section 2—obtain a diagrammatic expansion of Z to any order. The calculation of the dynamical correction—not explicitly reported here—is formally identical in the case when H is bilinear, and can be straightforwardly extended to higher order interactions.

5. *Conclusion*

It is worth noticing, in going through the previous example, that the sum over k in equation (4.42) is convergent only if

$$\beta \rangle \max_k \left[\frac{2 \coth^{-1} \mu_k}{E_k} \right] \tag{5.1}$$

This points out how the topology of the homogeneous space of the group is relevant in creating the singularities of Z. The particularly simple expression of Z as an integral over a manifold whose geometry is generally known from the group structure, should allow an *a priori* discussion of such singularities, without explicitly performing the calculation.

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